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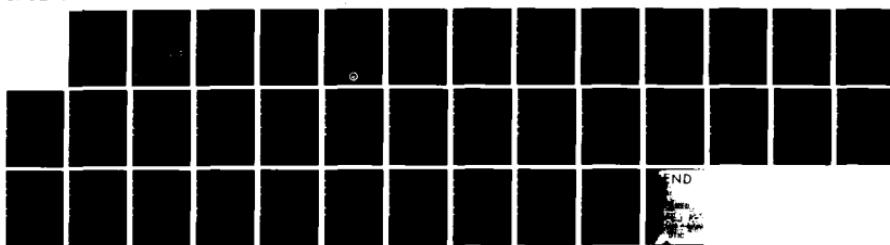
CONDITIONALLY HETEROSEDASTIC AUTOREGRESSIONS(U)  
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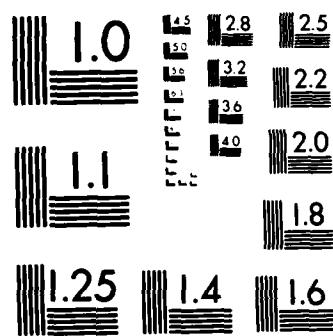
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# CONDITIONALLY HETROSCEDASTIC AUTOREGRESSIONS

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CONDITIONALLY HETROSCEDASTIC AUTOREGRESSIONS

Abbreviated Title:  
Heteroscedastic Autoregressions

by

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Key Words and Phrases: Nonlinear time series model,  
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process, stationary, ergodic, stationary initial  
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## ABSTRACT

Two conditionally heteroscedastic autoregressions are considered. It is shown that under suitable conditions, the processes are stationary and ergodic, and that the stationary initial distribution can be represented by a nonlinear function of independent, identically distributed standard Normal random variables.



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## 1. Introduction

Autoregressive moving average (ARMA) models are routinely used in time series analysis, particularly since the publication of Box and Jenkins' ( 1 ) book which set forth methods for the identification, estimation and diagnostic checking of these models. There is, however, no reason to suppose that every process can be reduced to one which is adequately represented by a linear model. Despite this, the linearity assumption is seldom questioned. The lack of useful, nonlinear alternatives to the ARMA model, with the exception of Granger and Andersen's ( 5 ) bilinear ARMA (BARMA) model, left few options. Recently, however, Engle ( 4 ) introduced a new class of nonlinear models. Engle ( 4 ) observed that, given the past, the conditional variance of a stochastic process is not necessarily constant but can, in general, depend on past observations. Accordingly, nonlinear, conditionally heteroscedastic time series models can be derived from the classical models. Although Engle only studied one specific formulation, conditional heteroscedasticity is a general property and can be used to define a variety of nonlinear models.

In this paper two conditionally heteroscedastic autoregressions are considered. Properties of the models are studied and simulated sample paths are presented. It is shown that under simple conditions the processes are stationary and ergodic, and that the stationary initial distribution

can be expressed in terms of independent, identically distributed (i.i.d.) standard Normal ( $N(0,1)$ ) random variables.

### 2 Conditionally Heteroscedastic Autoregressions

$\mathbb{E}^{\mathcal{F}_{t-1}} y_t = \phi_1 y_{t-1} + \dots + \phi_p y_{t-p} + \epsilon_t$  be a p-th order, zero mean autoregression and let  $\mathcal{J}_t = \sigma(y_t, y_{t-1}, \dots)$  be the sigma algebra generated by the  $y_t$ 's up to time t. In the classical model it is assumed that  $\{\epsilon_t\}$  is an i.i.d. sequence of  $N(0, \sigma_\epsilon^2)$  random variables, where  $\sigma_\epsilon^2 > 0$ . Consequently, given  $\mathcal{J}_{t-1}$ , the conditional mean of  $y_t$  is linear in  $y_{t-1}, \dots, y_{t-p}$  and the conditional variance is constant. Now, consider as an alternative model one where the conditional variance depends on the past observations. The result is the conditionally heteroscedastic autoregressive (CHAR) model:

$$y_t = \phi_1 y_{t-1} + \dots + \phi_p y_{t-p} + \epsilon_t \quad (2.1a)$$

where the distribution of  $\epsilon_t$ , conditioned on  $\mathcal{J}_{t-1}$ , is given by

$$\epsilon_t | \mathcal{J}_{t-1} \sim N(0, h_{t-1}) \quad (2.1b)$$

with  $h_{t-1} = h(y_{t-1}, y_{t-2}, \dots, y_{t-r}; \theta)$  for some finite  $r \geq 1$ .

The function  $h(\cdot)$  is a positive function of  $y_{t-1}, y_{t-2}, \dots, y_{t-r}$  and  $\theta$  is vector of parameters.

The CHAR model is obviously nonlinear, even though the

conditional mean is linear in  $y_{t-1}, y_{t-2}, \dots, y_{t-p}$ . In contrast, the nonlinearity of a BARMA model is the result of a nonlinear conditional mean. Engle (4) defined a general, conditionally heteroscedastic regression model in which the regressors can include exogenous as well as lagged variables. Most of the notation used here is Engle's and some of the properties which hold for the autoregression also hold for the more general regression model (see Engle (4)).

The CHAR process defined by (2.1a,b) is a Markov process where the order depends on  $h_{t-1}$  but is always at least p. For a given  $h_{t-1}$  it will be necessary to check whether or not the process is stationary. In any case, it is easy to see that  $E(\epsilon_t) = 0$ , for all t, and  $\text{Cov}(\epsilon_s, \epsilon_t) = 0$ , for  $s \neq t$ , since the conditional mean of  $\epsilon_t$  is zero. The  $\epsilon_t$ 's are uncorrelated although they obviously not independent. Thus, if the  $\{y_t\}$  process is stationary and has a finite variance, the autocorrelation function is the same as in the case of i.i.d.  $\epsilon_t$ 's. I.e., the autocorrelation at lag j,  $\rho_j$ , is given by

$$\rho_j = \text{Corr}(y_t, y_{t-j}) = A_1 G_1^j + \dots + A_p G_p^j \quad (2.2)$$

where  $A_1, \dots, A_p$  are constants and  $G_1^{-1}, \dots, G_p^{-1}$  are the roots of the characteristic polynomial  $\Phi(z) = 1 - \phi_1 z - \dots - \phi_p z^p$  with  $|G_i| < 1$ ,  $i = 1, 2, \dots, p$ . Furthermore, the unconditional vari-

ance,  $\sigma_y^2$ , is given by

$$\sigma_y^2 = E(h_{t-1})/(1-\phi_1\beta_1-\dots-\phi_p\beta_p) \quad (2.5)$$

if  $E(h_{t-1}) < \infty$ .

There are numerous choices for the conditional variance  $h_{t-1}$ . However, not all choices lead to useful models. It is desirable to choose  $h_{t-1}$  so that the model is flexible enough to give a good approximation to a variety of processes while remaining mathematically tractable. In this paper,  $h_{t-1}$  is assumed to be of the form  $h_{t-1} = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \dots + \alpha_k \epsilon_{t-k}^2$  or  $h_{t-1} = \beta_0 + \beta_1 (\phi_1 y_{t-1} + \dots + \phi_p y_{t-p})^2$ . The corresponding models will be referred to as CHARI and CHARII, respectively. The CHARI model is a special case of Engle's (4) conditionally heteroscedastic regression model and the CHARII model is the time series analog of the regression model in which the variance depends on the mean (see Carroll and Ruppert (3), for example). The properties of the CHARI and CHARII models are described in Sections 3 and 4. Proofs of the two theorems, which give sufficient conditions for the stationarity and ergodicity of the processes, are provided in Section 5.

### 3. The CHARI Model

The CHARI( $p, k$ ) model has the following representation:

$$\text{CHARI}(p, k): \quad y_t = \phi_1 y_{t-1} + \dots + \phi_p y_{t-p} + \epsilon_t \quad (3.1a)$$

$$\text{where } \epsilon_t = (\alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \dots + \alpha_k \epsilon_{t-k}^2)^{\frac{1}{2}} z_t \quad (3.1b)$$

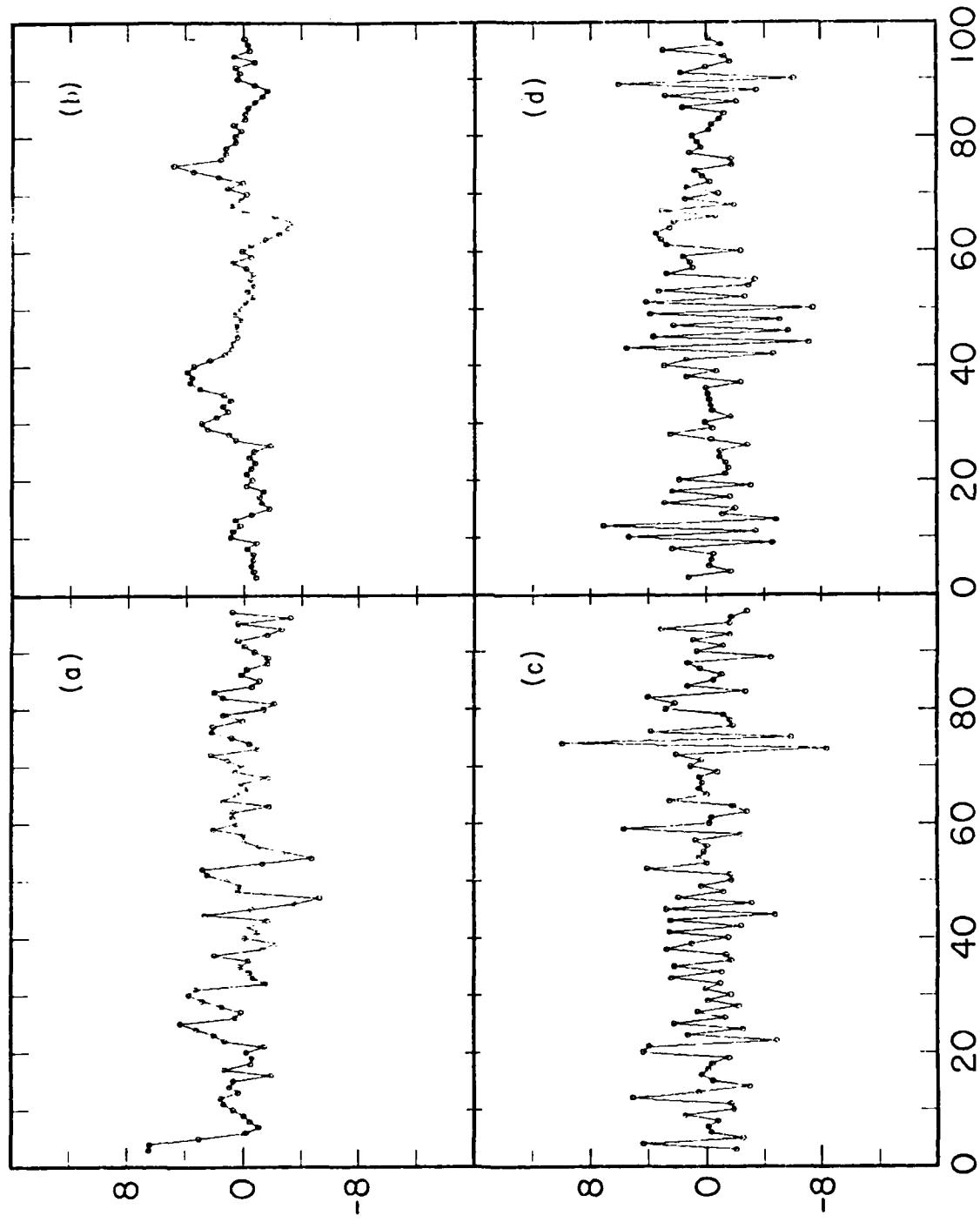
for some fixed, finite  $k > 0$  and  $\{z_t\} \sim \text{i.i.d. } N(0, 1)$ .

The parameters  $\alpha_0, \alpha_1, \dots, \alpha_k$  are nonnegative with  $\alpha_0$  strictly positive. Engle (4) referred to the  $\{\epsilon_t\}$  process, defined by (3.1b), as an "autoregressive-conditional-heteroscedastic" (ARCH) process. In that case the word "autoregressive" refers to the conditional variance structure and should not be confused with the autoregression in (3.1a) or the general CHAR process given by (2.1).

A CHARI( $p, k$ ) process is a nonlinear  $(p+k)$ -th order Markov process. Typical simulated sample paths are presented in Figure 1. These illustrate the "patchiness" which is characteristic of the CHARI process.

Conditions under which the process is stationary and ergodic have yet to be determined. It is obvious, however, that the roots of the characteristic polynomial  $\Phi(z)$  must lie outside the unit circle since this is required for the the special case where  $\alpha_1 = \alpha_2 = \dots = \alpha_k = 0$ , which corresponds to the usual autoregression. In the general model it will

Figure 1



also be necessary to impose restrictions on  $\alpha_1, \alpha_2, \dots, \alpha_k$  to ensure the stationarity of  $\{\epsilon_t\}$ . The appropriate condition holds when the roots of the characteristic polynomial  $\Psi(z) = 1 - \alpha_1 z - \dots - \alpha_k z^k$  lie outside the unit circle. Combining the two conditions gives the following theorem. A proof is provided in Section 5.

Theorem 3.1: (Stationarity and Ergodicity of the CHARI(p,k) Process) - Let  $\{y_t\}_{t=0}^\infty$  be a CHARI(p,k) process with  $\alpha_0 > 0$  and  $\alpha_i \geq 0$ ,  $i=1, 2, \dots, k$ . Assume that all the roots of both  $\Phi(z)$  and  $\Psi(z)$  lie outside the unit circle and,  $y_0, y_1, \dots, y_{p-1}$  and  $\epsilon_0, \epsilon_1, \dots, \epsilon_{k-1}$  have finite variances. Then  $\{y_t\}$  is asymptotically stationary and ergodic.

Under the conditions of Theorem 3.1, there is a unique stationary initial distribution. The corresponding stationary, joint density function for  $(y_{t-p-k+1}, y_{t-p-k+2}, \dots, y_t)^T$  is the nontrivial solution,  $f(y_{t-p-k+1}, \dots, y_t)$ , of the integral equation:

$$f(y_{t-p-k+1}, \dots, y_t) = \int_{-\infty}^{\infty} [2\Psi(\alpha_0 + \alpha_1 r_{t-1}^2 + \dots + \alpha_k r_{t-k}^2)]^{-\frac{1}{2}} \\ \times \exp[-\frac{1}{2}r_t^2/(\alpha_0 + \alpha_1 r_{t-1}^2 + \dots + \alpha_k r_{t-k}^2)] f(y_{t-p-k}, \dots, y_{t-1}) dy_{t-p-k} \quad (3.2)$$

where  $r_t = y_t - \phi_1 y_{t-1} - \dots - \phi_p y_{t-p}$ . The marginal distribution

of  $y_t$ , obtained by integration, can also be expressed as the distribution of a nonlinear function of i.i.d.  $N(0,1)$  random variables (see equation (5.13) in the proof of Theorem 3.1). Although the distribution of  $y_t$  is symmetric about zero, it is obviously non-Gaussian except when  $\alpha_1 = \alpha_2 = \dots = \alpha_k = 0$ .

Unfortunately, (3.2) appears to be difficult to solve, even in the simplest case with  $p=k=1$ . It is, however, easy to calculate the moments of  $\epsilon_t$  and  $y_t$  by using the conditional Normality to evaluate a conditional expectation first. Since the distribution of  $y_t$  is symmetric, all odd power moments exist and are zero. The unconditional variance of  $\epsilon_t$  can be found by solving the equation:

$$\sigma_\epsilon^2 = E(\epsilon_t^2) = E(h_{t-1}) = \alpha_0 + \alpha_1 E(\epsilon_{t-1}^2) + \dots + \alpha_k E(\epsilon_{t-k}^2).$$

Therefore,

$$\sigma_\epsilon^2 = \alpha_0 / (1 - \alpha_1 - \dots - \alpha_k) \quad (3.3)$$

where  $\sum_{i=1}^k \alpha_i < 1$  and substituting into (2.3) gives the stationary, unconditional variance of  $y_t$ :

$$\sigma_y^2 = \alpha_0 / [(1 - \alpha_1 - \dots - \alpha_k)(1 - \phi_1 \beta_1 - \dots - \phi_p \beta_p)] \quad (3.4)$$

where  $\sum_{i=1}^p \phi_i \beta_i < 1$ .

Higher order moments can be calculated in a similar

fashion, although this involves increasingly more algebra. In addition, the existence of higher order moments requires increasingly more stringent restrictions on the parameters, so that the only case where all moments are finite is the Gaussian ( $\alpha_1 = \alpha_2 = \dots = \alpha_k = 0$ ) case. For example, when  $p=k=1$ , (3.3) and (3.4) give

$$\sigma_\epsilon^2 = \alpha_0 / (1 - \alpha_1)$$

$$\sigma_y^2 = \alpha_0 / [(1 - \alpha_1)(1 - \phi^2)]$$

which are both finite and positive if  $\alpha_0 > 0$ ,  $0 < \alpha_1 < 1$  and  $| \phi | < 1$ . These are just the conditions for stationarity. However,

$$E(\epsilon_t^4) = E[E(\epsilon_t^4 | \mathcal{F}_{t-1})] = 3E[(\alpha_0 + \alpha_1 \epsilon_{t-1}^2)^2].$$

Therefore,

$$E(\epsilon_t^4) = 3\alpha_0^2(1 + \alpha_1^2) / [(1 - \alpha_1)(1 - 3\alpha_1^2)]$$

which is finite and positive only if  $0 < \alpha_1 < 3^{-\frac{1}{2}}$ .

In general, when  $p=k=1$ , the necessary and sufficient condition for the existence of  $E(\epsilon_t^{2j})$  is

$$\alpha_1^j \prod_{i=1}^j (2i-1) < 1, \quad j=1, 2, \dots \quad (3.5)$$

(Theorem 1; Engle (4) ). Since  $E(y_t^{2j}) > E(\epsilon_t^{2j})$ , (3.5) is necessary for the existence of  $E(y_t^{2j})$ . As  $j$  increases  $\alpha_1$  must decrease and in the limit,  $\alpha_1=0$  is the only case where all moments of  $y_t$  and  $\epsilon_t$  exist. Therefore, the density functions for  $y_t$  and  $\epsilon_t$  generally have heavier tails than the Normal distribution. Bounds for the tails can be obtained by determining the highest value of  $j$  for which  $E(y_t^{2j})$  or  $E(\epsilon_t^{2j})$  is finite.

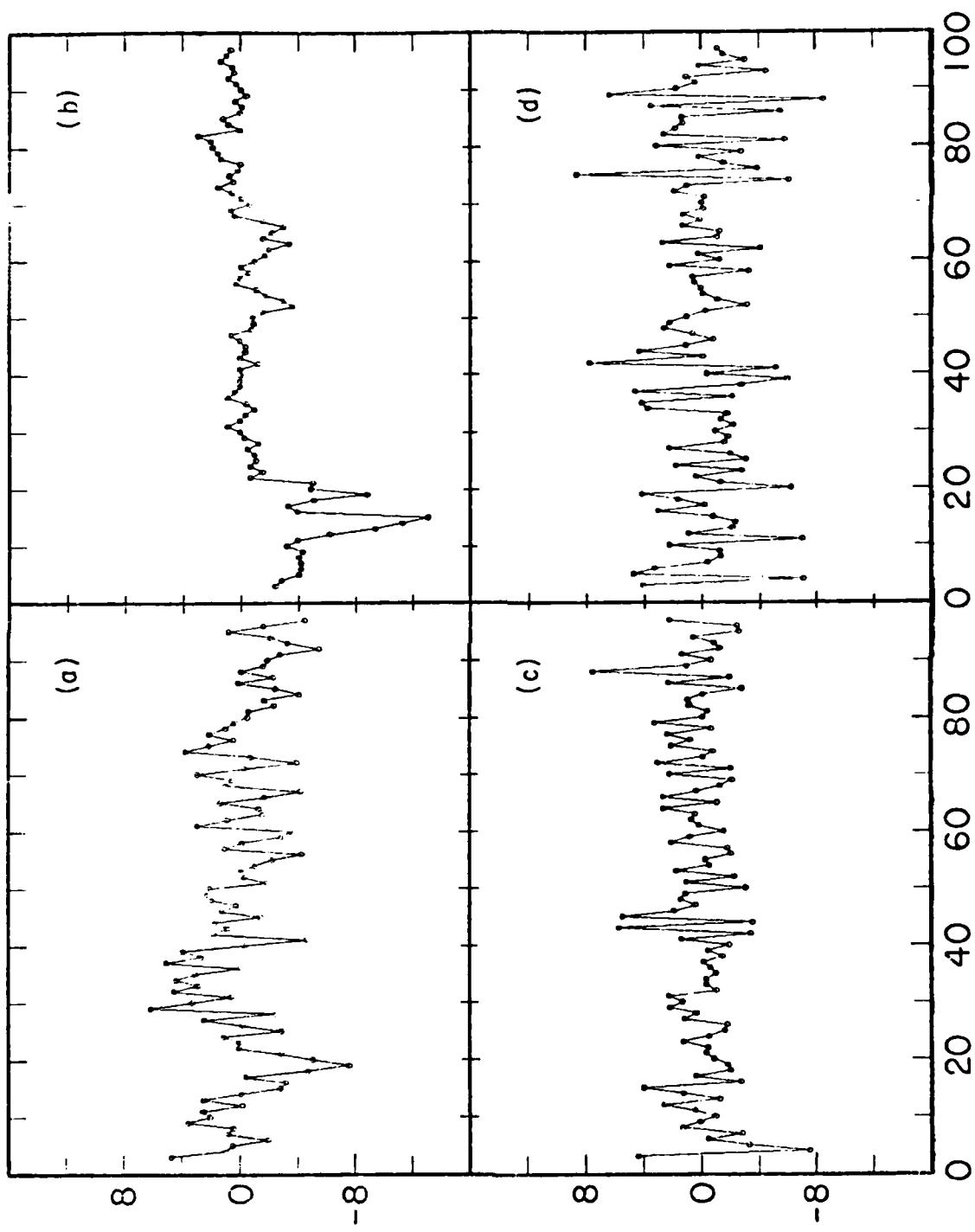
The CHARI model is an intuitively reasonable way of modelling conditional heteroscedasticity in an autoregression. Moreover, simulated sample paths, at least superficially, resemble certain time series in their patchy appearance. There are, however, other equally plausible ways of modelling the conditional variance of an autoregression. One obvious alternative is to assume that the conditional variance depends directly on the previous observations. In particular, the conditional variance may be directly related to the conditional mean. The second CHAR model, to be discussed in the next section, is one way of modelling such behaviour.

#### 4. The CHARI Model

The CHARI(p) model can be represented as follows:

$$\text{CHARII}(p): y_t = \phi_1 y_{t-1} + \dots + \phi_p y_{t-p} + \epsilon_t \quad (4.1a)$$

Figure 2



$$\text{where } \epsilon_t = [\beta_0 + \beta_1(\phi_1 y_{t-1} + \dots + \phi_p y_{t-p})^2]^{\frac{1}{2}} z_t \quad (4.1b)$$

and  $\{z_t\} \sim \text{i.i.d. } N(0,1)$ . The parameters  $\beta_0$  and  $\beta_1$  are non-negative with  $\beta_0$  strictly positive to ensure a positive conditional variance.

A CHARII process is simply the time series analog of the heteroscedastic regression model in which the variance is proportional to the square of the mean (see Carroll and Ruppert (3), for example). Like the CHARI process, the CHARII process is a nonlinear Markov process with the same linear conditional mean as the classical autoregression. Unlike the CHARI( $p,k$ ) process, the CHARII( $p$ ) has the same order,  $p$ , as the classical autoregression. Simulated sample paths, for four stationary CHARII processes, are shown in Figure 2.

Conditions which are sufficient for the stationarity and ergodicity of a CHARII( $p$ ) process are derived in Section 5 and are stated in Theorem 4.1. As before, it is obvious that the roots of the characteristic polynomial  $\Phi(z)$  must lie outside the unit circle for the process to be stationary. Less obvious is the requirement that  $(1+\beta_1)(\phi_1 \rho_1 + \dots + \phi_p \rho_p)$  be less than one, where  $\rho_j$ ,  $j=1,2,\dots,p$  are given by (2.2).

Theorem 4.1: (Stationarity and Ergodicity of the CHARII(p)

Process) - Let  $\{y_t\}_{t=0}^{\infty}$  be a CHARII(p) process with  $\beta_0 > 0$  and  $\beta_1 \geq 0$ . Assume that all the roots of  $\Phi(z)$  lie outside the unit circle and  $y_0, y_1, \dots, y_{p-1}$  have finite variances. Then  $\{y_t\}$  is asymptotically stationary and ergodic if  $(1 + \beta_1 + \phi_1 \rho_1 + \dots + \phi_p \rho_p) < 1$  where  $\rho_j$  is given by (2.2).

There is a unique stationary initial distribution when the conditions of Theorem 4.1 hold. The density function,  $f(y_{t-p+1}, \dots, y_t)$ , for the stationary initial distribution, can be found by solving:

$$f(y_{t-p+1}, \dots, y_t) = \int_{-\infty}^{\infty} [2\pi \{ \beta_0 + \beta_1 (\phi_1 y_{t-1} + \dots + \phi_p y_{t-p})^2 \}]^{-\frac{1}{2}} \\ \times \exp \left[ -\frac{1}{2} r_t^2 / \{ \beta_0 + \beta_1 (\phi_1 y_{t-1} + \dots + \phi_p y_{t-p})^2 \} \right] f(y_{t-p}, \dots, y_{t-1}) dy_{t-p} \quad (4.2)$$

where  $r_t = y_t - \beta_1 y_{t-1} - \dots - \phi_p y_{t-p}$ . The corresponding stationary, marginal distribution of  $y_t$  can be represented in terms of a nonlinear function of i.i.d.  $N(0,1)$  random variables (see Equation (5.23) in the next section).

The integral equation (4.2) is just as difficult to solve as (3.2). However, moments can be calculated by invoking the conditional Normality. For example, the variance,  $\sigma_y^2$ , of a stationary CHARII(p) process can be evaluated by substituting

$$\begin{aligned} E(h_{t-1}) &= \beta_0 + \beta_1 E[(\phi_1 y_{t-1} + \dots + \phi_p y_{t-p})^2] \\ &= \beta_0 + \beta_1 (\phi_1 \beta_1 + \dots + \phi_p \beta_p) \sigma_y^2 \end{aligned}$$

into (2.3) and solving for  $\sigma_y^2$  to give

$$\sigma_y^2 = \beta_0 / [1 - (1 + \beta_1)(\phi_1 \beta_1 + \dots + \phi_p \beta_p)] \quad (4.3)$$

where  $(1 + \beta_1)(\phi_1 \beta_1 + \dots + \phi_p \beta_p) < 1$ .

In general, the marginal distribution of  $y_t$  is symmetric, but not Gaussian, with heavier tails than Gaussian. All odd moments are zero, by symmetry, and  $E(y_t^{2j})$  exists and is finite for a finite number of  $j$ , unless  $\beta_1 = 0$  where all moments exist. The maximum value of  $j$  for which  $E(y_t^{2j})$  is finite can be used to bound the tails of the distribution of  $y_t$ . On the other hand, if  $E(y_t^{2j})$  is finite, for a given  $j$ , then certain restrictions on the parameters are implied. For example, when  $p=1$ ,  $E(y_t^{2j}) < \infty$  implies that

$$1 - \phi^{2j} \sum_{i=1}^{2j} \binom{2j}{2i} \prod_{m=1}^{j-i} (2m-1) \beta_1^{j-i} > 0. \quad (4.4)$$

The corresponding regions of the parameter space are shown in Figure 3, for  $j=1$ , which implies stationarity, and for  $j=2$ , which corresponds to a finite fourth moment given by

$$E(y_t^4) = [3\beta_0^2 + 6\phi^2\beta_0\sigma_y^2(1+\beta_1)] / [1 - \phi^4(1+6\beta_1+3\beta_1^2)] \quad (4.5)$$

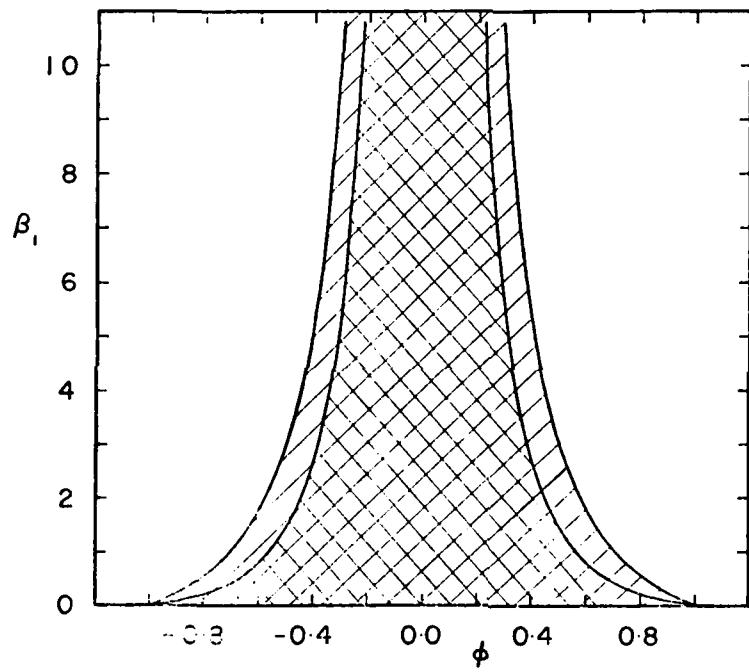


Figure 3

When  $j=2$ , condition (4.4) ensures that the denominator of (4.5) is finite and positive.

Theorems 3.1 and 4.1 are proved in the next section. The proofs are important because they establish the conditions under which the CHARI and CHARII processes are stationary and ergodic and, in doing so, characterize the stationary initial distribution. Stationarity and ergodicity are fundamental properties. In practical applications, stationarity is often imposed by physical constraints on the process producing the observations. In some cases, stationarity is required if the model is to have a meaningful interpretation. Furthermore, the assumption of stationarity and ergodicity is central to statistical inference, including parameter estimation. Theorems 3.1 and 4.1 are essential to the investigation and application of the CHARI and CHARII models.

### 5. Proofs of Theorems 3.1 and 4.1

Both Theorems 3.1 and 4.1 are proved using the results of Breiman (2) which demonstrate that the stationarity and ergodicity of a Markov process depend on the existence and uniqueness of a stationary initial distribution. It will be shown, by repeated application of the appropriate defining equation, (3.1a,b) or (4.1a,b), that as  $t \rightarrow \infty$ , the limiting distribution of  $y_t$  exists and is independent of the initial conditions. This distribution is the unique

stationary initial distribution and the process which has it as its initial distribution is stationary and ergodic (Theorem 7.16, Breiman (2)). In this sense, the CHARI and CHARII processes are asymptotically stationary and ergodic, if the conditions of Theorem 3.1 or 4.1 hold.

### Proof of Theorem 3.1

Consider the process  $\{\epsilon_t\}$  defined by (3.1b). Repeated application of (3.1b) gives:

$$\epsilon_k = (\alpha_0 + \alpha_1 \epsilon_{k-1}^2 + \dots + \alpha_k \epsilon_0^2)^{\frac{1}{2}} z_k$$

$$\epsilon_{k+1} = [\alpha_0 + \alpha_1 (\alpha_0 + \alpha_1 \epsilon_{k-1}^2 + \dots + \alpha_k \epsilon_0^2) z_k^2 + \alpha_2 \epsilon_{k-1}^2 + \dots + \alpha_k \epsilon_1^2]^{\frac{1}{2}} z_{k+1}$$

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$$\epsilon_{k+j} = \left[ \sum_{m=0}^j u_m + v_{j,1} \epsilon_{k-1}^2 + \dots + v_{j,k} \epsilon_0^2 \right]^{\frac{1}{2}} z_{k+j} \quad j=0, 1, \dots \quad (5.1)$$

where the random variables  $u_m$  and  $v_{j,m}$  are defined recursively by the following relations:

$$u_0 = \alpha_0$$

$$u_1 = \alpha_1 u_0 z_k^2$$

.

$$u_k = \alpha_1 u_{k-1} z_{2k-1}^2 + \alpha_2 u_{k-2} z_{2k-2}^2 + \dots + \alpha_k u_0 z_k^2$$

.

(5.2)

$$u_j = \alpha_1 u_{j-1} z_{k+j-1}^2 + \alpha_2 u_{j-2} z_{k+j-2}^2 + \dots + \alpha_k u_{j-k} z_j^2 \quad j > k \quad \boxed{}$$

and for  $m=1, 2, \dots, k$

$$v_{0,m} = \epsilon_m$$

$$v_{1,m} = \alpha_1 v_{0,m} z_k^2$$

$\vdots$

$$v_{k,m} = \alpha_1 v_{k-1,m} z_{2k-1}^2 + \alpha_2 v_{k-2,m} z_{2k-2}^2 + \dots + \alpha_k v_{0,m} z_k^2$$

$\vdots$

$\vdots$

$$v_{j,m} = \alpha_1 v_{j-1,m} z_{k+j-1}^2 + \alpha_2 v_{j-2,m} z_{k+j-2}^2 + \dots + \alpha_k v_{j-k,m} z_j^2 \quad j > k \quad \boxed{(5.3)}$$

Equation (5.1), which can be verified by induction on  $j$ , expresses  $\epsilon_{k+j}$  in terms of the initial values  $\epsilon_0, \epsilon_1, \dots, \epsilon_{k-1}$  and the i.i.d.  $N(0,1)$  random variables  $z_k, z_{k+1}, \dots, z_{k+j}$ , for  $j=0, 1, \dots$ .

Note that the random variables  $v_{j,m}$ ,  $j=1, 2, \dots, k$  are nonnegative for all  $j$  and from (5.3) it follows that for  $m=1, 2, \dots, k$

$$E(v_{0,m}) = \alpha_m$$

$$E(v_{1,m}) = \alpha_1 E(v_{0,m})$$

$\vdots$

$$E(v_{j,m}) = \alpha_1 E(v_{j-1,m}) + \alpha_2 E(v_{j-2,m}) + \dots + \alpha_k E(v_{j-k,m}) \quad (5.4)$$

where (5.4) holds for  $j \geq k$  and  $E(v_{j,m})$ ,  $j=0, 1, \dots, k-1$ , are obviously finite. Equation (5.4) is a difference equation in  $E(v_{j,m})$ , for each  $m$ , and has the solution

$$E(v_{j,m}) = B_{1,m} F_1^j + \dots + B_{k,m} F_k^j \quad (5.5)$$

where  $B_{1,m}, \dots, B_{k,m}$  are constants and  $F_1^{-1}, F_2^{-1}, \dots, F_k^{-1}$  are the roots of the characteristic polynomial

$$\Psi(z) = 1 - \alpha_1 z - \dots - \alpha_k z^k,$$

which lie outside the unit circle by hypothesis. From (5.5) the expected values,  $E(v_{j,m})$ , die out exponentially with  $j$  and hence  $\sum_{j=0}^{\infty} E(v_{j,m}) < \infty$  for  $m=1, 2, \dots, k$ . This implies that

$$\sum_{j=0}^{\infty} \sum_{m=1}^k E(v_{j,m} \epsilon_{k-m}^2) = \sum_{j=0}^{\infty} \sum_{m=1}^k E(v_{j,m}) E(\epsilon_{k-m}^2) < \infty \quad (5.6)$$

since the  $v_{j,m}$ , which depend on  $z_k, z_{k+1}, \dots, z_j$ , are independent of  $\epsilon_0, \epsilon_1, \dots, \epsilon_{k-1}$ , and by hypothesis  $\epsilon_0, \epsilon_1, \dots, \epsilon_{k-1}$  have finite second moments.

Now consider the CHARI( $p, k$ ) process given by (3.1a, b). A standard argument for the  $p$ -th order autoregression can be applied to (3.1a) (see Karlin and Taylor (6), for example). This gives

$$y_{p+j} = \sum_{m=0}^j \xi_m \epsilon_{p+j-m} + \xi_{j,1} y_{p-1} + \dots + \xi_{j,p} y_0 \quad j=0, 1, \dots \quad (5.7)$$

where the recurrence relations for  $\xi_j$  and  $\xi_{j,m}$  are:

$$\begin{aligned} \xi_0 &= 1 \\ \xi_{j+1} &= \xi_{j,1} \quad j=0, 1, \dots \end{aligned} \quad (5.8)$$

$$\xi_{0,m} = \phi_m \quad m=1, 2, \dots, p$$

$$\xi_{j+1,m} = \xi_{j,1} \phi_m + \xi_{j,m+1} \quad m=1, 2, \dots, p-1 \quad j=0, 1, \dots \quad (5.9)$$

$$\xi_{j,p} = \xi_{j,1} \phi_p.$$

The coefficients can also be obtained from the power series expansion of  $\Phi^{-1}(z)$  where

$$\Phi(z) = 1 - \phi_1 z - \dots - \phi_p z^p.$$

In particular,

$$\Phi^{-1}(z) = \sum_{m=0}^j \xi_m z^m + R_j(z) \quad (5.10)$$

where the remainder  $R_j(z)$  is

$$R_j(z) = (\xi_{j,1}z^{j+1} + \dots + \xi_{j,p}z^{p+j})\Phi^{-1}(z). \quad (5.11)$$

Furthermore, if  $G_1^{-1}, \dots, G_p^{-1}$  are the roots of  $\Phi(z)$ , which lie outside the unit circle by assumption, and  $|z| < \min_i |G_i^{-1}|$ , then the series  $\tilde{\Phi}^{-1}(z) = \sum_{m=0}^{\infty} \delta_m z^m$  converges absolutely. Hence  $R_j(z) \rightarrow 0$ , as  $j \rightarrow \infty$ , for any  $|z| < \min_i |G_i^{-1}|$ , including  $z=1$ . This implies that  $\xi_{j,m} \rightarrow 0$  for  $m=1, 2, \dots, p$ , as  $j \rightarrow \infty$ , and  $\sum_{m=0}^{\infty} \delta_m$  converges absolutely (see Karlin and Taylor (6) for details).

Now, combining (5.1) and (5.7) gives

$$\begin{aligned} & E \left[ \left\{ y_{p+j} - \sum_{m=0}^j \delta_m \left( \sum_{i=0}^{p+j-m} u_i \right)^{\frac{1}{2}} z_{p+j-m} \right\}^2 \right] \\ &= E[(\xi_{j,1}y_{p-1} + \dots + \xi_{j,p}y_0)^2] + \\ & \quad \sum_{m=0}^j \delta_m^2 E \left[ \left\{ \left( \sum_i u_i + v_{p+j-m-k,1}\epsilon_{k-1}^2 + \dots + v_{p+j-m-k,k}\epsilon_0^2 \right)^{\frac{1}{2}} - \left( \sum_i u_i \right)^{\frac{1}{2}} \right\}^2 \right] \\ &\leq E[(\xi_{j,1}y_{p-1} + \dots + \xi_{j,p}y_0)^2] + \\ & \quad \sum_{m=0}^j \delta_m^2 E(v_{p+j-m-k,1}\epsilon_{k-1}^2 + \dots + v_{p+j-m-k,k}\epsilon_0^2). \end{aligned} \quad (5.12)$$

The first term in the last expression vanishes, as  $j \rightarrow \infty$ , if  $y_0, y_1, \dots, y_{p-1}$  have finite variances, since  $\xi_{j,m} \rightarrow 0$ ,  $m=1, 2, \dots, p$ . The last term is the  $(j+1)$ -st term of the Cauchy product of  $\sum_{m=0}^{\infty} \delta_m^2$  and  $\sum_{j=0}^{\infty} E(v_{p+j-k,1}\epsilon_{k-1}^2 + \dots + v_{p+j-k,k}\epsilon_0^2)$  and therefore must also converge to zero since the Cauchy product converges, as a

consequence of  $\sum_{m=0}^{\infty} |s_m| < \infty$  and (5.6). Hence, the expectation (5.12) converges to zero as  $j \rightarrow \infty$ .

The preceding results imply that, as  $j \rightarrow \infty$ ,

$$y_{p+j} \rightarrow U$$

in quadratic mean, where

$$U = \lim_{j \rightarrow \infty} \sum_{m=0}^j s_m \left( \sum_i u_i \right)^{\frac{1}{2}} z_{p+j-m}. \quad (5.13)$$

The distribution of  $U$  is independent of the distribution of  $y_0, y_1, \dots, y_{p-1}$  and  $\epsilon_0, \epsilon_1, \dots, \epsilon_{k-1}$ . Since the  $\{y_t\}$  process is obviously indecomposable, the limiting distribution, given by (5.13), is the unique stationary initial distribution and the  $\{y_t\}$  process is asymptotically stationary and ergodic (Breiman (2)).

#### Proof of Theorem 4.1

Repeated substitution of (4.1a, b) gives:

$$y_p = \phi_1 y_{p-1} + \dots + \phi_p y_0 + [\beta_0 + \beta_1 (\phi_1 y_{p-1} + \dots + \phi_p y_0)^2]^{\frac{1}{2}} z_p$$

$$\begin{aligned}
 y_{p+1} &= \phi_1 \{ \phi_1 y_{p-1} + \dots + \phi_p y_0 + [\beta_0 + \beta_1 (\phi_1 y_{p-1} + \dots + \phi_p y_0)^2]^{\frac{1}{2}} z_p \} \\
 &\quad + \phi_2 y_{p-1} + \phi_3 y_{p-2} + \dots + \phi_p y_1 + [\beta_0 + \beta_1 (\phi_1 y_{p-1} + \dots \\
 &\quad - \phi_p y_0 + [\beta_0 + \beta_1 (\phi_1 y_{p-1} + \dots + \phi_p y_0)^2]^{\frac{1}{2}} z_p] + \phi_2 y_{p-1} + \dots + \phi_p y_1]^{\frac{1}{2}} z_{p+1} \\
 &\quad \vdots \\
 y_{p+j} &= \sum_{m=0}^j \delta_m w_{j-m} + \xi_{j,1} y_{p-1} + \dots + \xi_{j,p} y_0 \quad j=0, 1, \dots \quad (5.14)
 \end{aligned}$$

where the coefficients  $\delta_m$ ,  $m=0, 1, \dots$  and  $\xi_{j,m}$ ,  $m=1, 2, \dots, p$  are defined by the recurrence relations (5.8) and (5.9) of the previous proof.

The random variables  $w_m$ ,  $m=0, 1, \dots$ , are also defined recursively. They are given by

$$\begin{aligned}
 w_0 &= [\beta_0 + \beta_1 (\phi_1 y_{p-1} + \dots + \phi_p y_0)^2]^{\frac{1}{2}} z_p \\
 w_{j+1} &= \{ \beta_0 + \beta_1 (\xi_{j+1,1} y_{p-1} + \dots + \xi_{j+1,p} y_0 + \delta_{j+1} w_0 + \dots + \delta_1 w_j)^2 \}^{\frac{1}{2}} z_{p+j+1}, \\
 &\quad j=0, 1, \dots \quad (5.15)
 \end{aligned}$$

The  $w_j$ 's are uncorrelated and have mean zero. The variance of  $w_{j+1}$  can be found by squaring both sides of (5.15) and taking expectations to give

$$E(w_{j+1}^2) = \beta_0 + \beta_1 E[(\xi_{j+1,1}y_{p-1} + \dots + \xi_{j+1,1}y_0)^2] +$$

$$\delta_{j+1}^2 E(w_0^2) + \dots + \delta_1^2 E(w_j^2)$$

$$E(w_j^2) = \beta_0 + \beta_1 E[(\phi_1 y_{p-1} + \dots + \phi_p y_0)^2]. \quad \text{Therefore,}$$

$$E(w_j^2) = \beta_0 \sum_{m=0}^j \eta_m + \beta_1 \sum_{m=0}^j \eta_m \sigma_{j-m}^2 \quad j=0, 1, \dots \quad (5.16)$$

where  $\sigma_j^2$  is defined to be

$$\sigma_j^2 = E[(\xi_{j,1}y_{p-1} + \dots + \xi_{j,p}y_0)^2] \quad j=0, 1, \dots$$

and the nonnegative coefficients  $\eta_j$  are given by the recurrence relations

$$\eta_0 = 1$$

$$\eta_j = \beta_1 (\delta_1^2 \eta_{j-1} + \dots + \delta_j^2 \eta_0) \quad j=1, 2, \dots . \quad (5.17)$$

Consider the power series  $\sum_{m=0}^{\infty} \eta_m z^m$ . Equation (5.17) implies that

$$(\sum_{m=0}^{\infty} \eta_m z^m)(1 + \beta_1 - \beta_1 \sum_{m=0}^{\infty} \delta_m^2 z^m) = 1. \quad (5.18)$$

Since  $\sum_{m=0}^{\infty} \delta_m^2 z^m$  is absolutely convergent whenever  $|z| < \min_i |G_i^{-\frac{1}{2}}|$ ,

where  $G_i^{-1}$ ,  $i=1, 2, \dots, p$  are the roots of  $\Phi(z)$  (see previous proof), (5.18) implies that  $\sum_{m=0}^{\infty} \eta_m z^m$  is absolutely convergent if  $|z| < \min_i |G_i^{-1}|$  and  $1 + \beta_1 - \beta_1 \sum_{m=0}^{\infty} \delta_m^2 |z|^m > 0$ . In particular, for  $z=1$ ,  $\sum_{m=0}^{\infty} \eta_m < \infty$  if  $1 + \beta_1 - \beta_1 \sum_{m=0}^{\infty} \delta_m^2 > 0$ . Note that  $1 + \beta_1 - \beta_1 \sum_{m=0}^{\infty} \delta_m^2 > 0$  is equivalent to  $(1 + \beta_1)(\phi_1 \rho_1 + \dots + \phi_p \rho_p) < 1$  since  $\sum_{m=0}^{\infty} \delta_m^2 = (1 - \phi_1 \rho_1 - \dots - \phi_p \rho_p)^{-1}$  where  $\rho_1, \dots, \rho_p$  are given by (2.2).

It is also easy to see that  $\sum_{m=0}^{\infty} \sigma_m^2 < \infty$ . First note that the coefficients  $\xi_{j,m}$ ,  $m=1, 2, \dots, p$ , satisfy  $\sum_{j=0}^{\infty} R_j(z) = \sum_{m=0}^{\infty} m \delta_m z^m$ , where the remainder was defined by (5.11), and  $\sum_{m=0}^{\infty} m \delta_m z^m$  is absolutely convergent for  $|z| < \min_i |G_i^{-1}|$ . Thus, for  $z=1$ ,  $\sum_{j=0}^{\infty} |(\xi_{j,1} + \dots + \xi_{j,p})| < \infty$  which implies that  $\sum_{j=0}^{\infty} \sigma_j^2 < \infty$ , provided  $y_0, y_1, \dots, y_{p-1}$  have finite variances.

Let  $\{x_j\}$  be the random variables defined by

$$x_0 = \beta_0^{\frac{1}{2}} z_p$$

$$x_{j+1} = [\beta_0 + \beta_1 (\delta_{j+1} x_0 + \dots + \delta_1 x_j)^2]^{\frac{1}{2}} z_{p+j+1} \quad j=0, 1, \dots .$$

Repeating the argument used to derive (5.16),

$$E(x_j^2) = \beta_0 \sum_{m=0}^j \eta_m . \quad (5.19)$$

Now, consider  $E[(y_{p+j} - \sum_{m=0}^j \delta_m x_{j-m})^2]$ . From (5.14)

$$E[(y_{p+j} - \sum_{m=0}^j \delta_m x_{j-m})^2] = \sigma_j^2 + \sum_{m=0}^j \delta_m^2 E[(w_{j-m} - x_{j-m})^2]. \quad (5.20)$$

The expectation of  $w_j x_j$  is nonnegative and

$$\begin{aligned} E(w_j x_j) &\geq \beta_0 + \beta_1 E\left[\left\{\left(\xi_{j,1} y_{p-1} + \dots + \xi_{j,p} y_0 + \delta_j w_0 + \dots + \delta_1 w_{j-1}\right)^2\right.\right. \\ &\quad \left.\left. + (\delta_j x_0 + \dots + \delta_1 x_{j-1})^2\right\}^{\frac{1}{2}}\right]. \end{aligned} \quad (5.21)$$

The inequality follows from the fact that  $(\beta_0 + \beta_1 a^2)^{\frac{1}{2}} (\beta_0 + \beta_1 b^2)^{\frac{1}{2}}$   $\geq \beta_0 + \beta_1 (a^2 b^2)^{\frac{1}{2}}$  for any real numbers  $a$  and  $b$ . Since  $E[(X^2 Y^2)^{\frac{1}{2}}] = E(|XY|) \geq |E(XY)|$  for random variables  $X$  and  $Y$ , (5.21) implies that

$$E(w_j x_j) \geq \beta_0 + \beta_1 [\delta_j^2 E(w_0 x_0) + \dots + \delta_1^2 E(w_{j-1} x_{j-1})].$$

This inequality and the fact that

$$E(w_0 x_0) = E\left[\beta_0^{\frac{1}{2}} \left\{ \beta_0 + \beta_1 (\phi_1 y_{p-1} + \dots + \phi_p y_0)^2 \right\}^{\frac{1}{2}}\right] \geq \beta_0$$

can be used to verify that  $E(w_j x_j) \geq \beta_0 \sum_{m=0}^j \eta_m$ . Hence it follows from (5.15) and (5.19) that

$$E[(w_j - x_j)^2] \leq \beta_1 \sum_{m=0}^j \eta_m \sigma_{j-m}^2.$$

Substituting into (5.20) and simplifying, using (5.17), gives

$$E[(y_{p+j} - \sum_{m=0}^j \delta_m x_{j-m})^2] \leq (1+\beta_1) \sum_{m=0}^j \eta_m \sigma_{j-m}^2. \quad (5.22)$$

The right-hand side of (5.22) converges to zero as  $j \rightarrow \infty$ , since  $\sum_{m=0}^{\infty} \eta_m$  and  $\sum_{m=0}^{\infty} \sigma_m^2$  both converge and the right-hand side of (5.22) is the  $(j+1)$ -st term of the Cauchy product of the two series.

The foregoing arguments imply that, as  $j \rightarrow \infty$ ,

$$y_{p+j} \rightarrow x$$

in quadratic mean, where

$$x = \lim_{j \rightarrow \infty} \sum_{m=0}^j \delta_m x_{j-m}. \quad (5.23)$$

The distribution of  $X$  corresponds to the unique stationary initial distribution and does not depend on  $y_0, y_1, \dots, y_{p-1}$ . The CHARII( $p$ ) process is, therefore, asymptotically stationary and ergodic Breiman (2) ).

## 6. Conclusions

The CHARI and CHARII models are two representatives of a general class of conditionally heteroscedastic autoregressions. Simulated sample paths, generated by the CHARI and CHARII processes, have a characteristic patchy appearance. The processes possess several attractive properties, such

as a linear conditional mean and an autocorrelation function which is the same as the autocorrelation function of the classical autoregression. In addition, the processes can be shown to be stationary and ergodic under a reasonable and easily verifiable set of assumptions.

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### Figure Captions

Figure 1: Simulated sample paths of four stationary CHARI processes. The mean and variance are the same for all four cases.

- (a) CHARI(1,1)  $\phi=.5$ ;  $\alpha_0=1$ ,  $\alpha_1=.9$
- (b) CHARI(1,1)  $\phi=.9$ ;  $\alpha_0=1$ ,  $\alpha_1=.9$
- (c) CHARI(2,1)  $\phi_1=-.4$ ,  $\phi_2=.2$ ;  $\alpha_0=1.92$ ,  $\alpha_1=.8$
- (d) CHARI(2,2)  $\phi_1=-.4$ ,  $\phi_2=.2$ ;  $\alpha_0=1.92$ ,  $\alpha_1=.4$ ,  $\alpha_2=.4$

Figure 2: Simulated sample paths of four stationary CHARII processes. The mean and variance are the same for all four cases.

- (a) CHARII(1)  $\phi=.5$ ;  $\beta_0=.7$ ,  $\beta_1=.9$
- (b) CHARII(1)  $\phi=.9$ ;  $\beta_0=.3733$ ,  $\beta_1=.2$
- (c) CHARII(2)  $\phi=-.4$ ,  $\phi_2=.2$ ;  $\beta_0=2.1333$ ,  $\beta_1=2$
- (d) CHARII(2)  $\phi=-.4$ ,  $\phi_2=.2$ ;  $\beta_0=5.8666$ ,  $\beta_1=1$

Figure 3: The regions of stationarity and finite fourth moment for the CHARII(1) process. The region between the two outer curves is the region of stationarity:  $(1+\beta_1)\phi^2 \leq 1$  and the region between the inner curves is the region where the fourth moment exists:  $\phi^4(1+6\beta_1+3\beta_1^2) \leq 1$ .

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